Limiting the Complexity of Sphere Decoding for UW-OFDM

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***Abstract*— The recently presented UW-OFDM (unique word - orthogonal frequency division multiplexing) signaling scheme [1] uses certain subcarriers in frequency domain for calculated symbols instead of data, in order to introduce redundancy. The resulting linear OFDM system model can be interpreted as a fading multiple antenna system, which enables sophisticated detectors for data recovery, known for MIMO (multiple input multiple output) channels. In [2] we implemented the Sphere Decoder (SD) for a single antenna UW-OFDM system as a realization of the optimum maximum likelihood sequence estimator (MLSE). However, the SD's execution time can vary extensively, which makes it inapplicable for practical systems. In order to limit its computation time, the number of nodes visited during the SD tree search can be limited [3], degrading its performance though. In this work we investigate the impact of limiting the number of node visits on the bit error performance. We show that at low E_b/N_0 values, and especially in flat fading environments the SD tends to exhaustive searches. In multi-path environments however, the UW-OFDM system structure together with the channel matrix yield a friendly environment, that leads to an early termination of the SD with high probability.**

***Index Terms*— OFDM, Unique Word, Sphere Decoding, Complexity, Node visits**

I. Introduction

THE recently presented Unique Word (UW) OFDM signaling scheme [1] uses a deterministic sequence in the guard interval of an OFDM symbol, instead of the cyclic prefix as it is used in most current OFDM applications. By putting the UW inside the DFT (discrete Fourier transform) interval in time domain, this ensures the usual cyclicity of the OFDM symbol. This is achieved by loading certain subcarriers in frequency domain with redundant symbols instead of data, which solely depend on the data, and thus introduce correlation.

For the classic and well known cyclic prefix OFDM, equalizers following the zero forcing (ZF) principle are optimal. On the contrary, UW-OFDM allows for more elaborate data estimation, taking correlations into account, which are introduced by the UW-OFDM structure. Initially introduced in [1], the LMMSE (linear minimum mean square error) data estimator uses some information from the redundant carriers beneficially in order to improve the bit error rate (BER). However, the fact that correlations are introduced as part of the transmit process allows the use of many more receiver structures, e.g. all those known from the MIMO (multiple input multiple output) world.

As the best possible receiver for this setup executes a maximum likelihood sequence estimation (MLSE) on a whole OFDM symbol, we used the practically feasible Sphere Decoding (SD) algorithm to achieve this performance for UW-OFDM [2], [4]. However, the computational complexity of the SD, which is basically a tree-search, can get out of hand. Depending on the current channel and noise conditions, the SD might work a majority of the tree branches, offering no advantage over a brute force MLSE, i.e. checking every leaf of the tree. An effective countermeasure is the introduction of an upper limit for the number of nodes to visit. This is called the constrained SD [3]. Of course, it will effect the performance of the Sphere Decoder, which is investigated and evaluated for UW-OFDM systems in this work.

The paper is organized as follows: A brief UW-OFDM system description will be presented in Sec. II. The Sphere Decoding principle will be revised in Sec. III. We investigate the effect of limiting the number of node visits in different channels in Sec. IV and show simulation results that quantify the gain of Sphere Decoding in comparison to the LMMSE estimator. Sec. V concludes this work.

II. Unique Word OFDM System Model

We briefly review the approach of introducing unique words in OFDM. For further details see [1].

Let $\mathbf{x}_u$ be a predefined sequence of length N_u, which we call unique word. This unique word shall form the tail of the OFDM time domain symbol vector. Hence, the time domain symbol vector, as the result of the length N IDFT (inverse DFT), consists of two parts and is of the form $\begin{bmatrix}\mathbf{x}_d^\mathsf{T} & \mathbf{x}_u^\mathsf{T}\end{bmatrix}^\mathsf{T} \in \mathbb{C}^{N\times 1}$, whereas only $\mathbf{x}_d \in \mathbb{C}^{(N-N_u)\times 1}$ is random and affected by the data. In order to generate the OFDM symbol, it turned out that it is advantageous to generate an OFDM symbol

$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_d \\ \mathbf{0}\end{bmatrix} \tag{1}$$

with a zero UW in a first step, and to add the desired UW to determine the transmit symbol $\mathbf{x}' = \mathbf{x} + \begin{bmatrix}\mathbf{0}^\mathsf{T} & \mathbf{x}_u^\mathsf{T}\end{bmatrix}^\mathsf{T}$ in a second step [5]. As we intend N_u zero symbols acc. to (1) as a part of the IDFT output, we have to load at least the same number of subcarriers in frequency domain with appropriate values. We spend $N_r = N_u$ subcarriers and choose to name them *redundant* subcarriers, as they are loaded with values depending on the data $\tilde{\mathbf{d}} \in \mathbb{C}^{N_d\times 1}$.

Following the derivations in [1] and employing the N point DFT matrix $\mathbf{F}_N$ with its element in the k-th row and the l-th column $[\mathbf{F}_N]_{kl} = \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kl}$, the transmit symbol can be written as

$$\mathbf{x}' = \mathbf{F}_N^{-1}\mathbf{G}\tilde{\mathbf{d}} + \begin{bmatrix}\mathbf{0} \\ \mathbf{x}_u\end{bmatrix}, \tag{2}$$

utilizing the UW-OFDM symbol generator matrix $\mathbf{G} \in \mathbb{C}^{N\times N_d}$. Transforming the UW into frequency domain $\tilde{\mathbf{x}}_u = \mathbf{F}_N^{-1}\begin{bmatrix}\mathbf{0}^\mathsf{T} & \mathbf{x}_u^\mathsf{T}\end{bmatrix}^T$ allows us to rewrite (2) as

$$\mathbf{x}' = \mathbf{F}_N^{-1}(\mathbf{G}\tilde{\mathbf{d}} + \tilde{\mathbf{x}}_u). \tag{3}$$

Please note, that in contrast to [1], [5], we do not introduce zero subcarriers in frequency domain at the band edges and at the DC carrier. The propagation of the OFDM symbol, assembled according to (3), through a multi-path channel is modeled using a convolution matrix $\mathbf{H}_c$ and a noise vector $\mathbf{n}$ with the covariance matrix $\mathbf{C}_{nn} = \sigma_n^2\mathbf{I}$. After applying the DFT in the receiver, the frequency domain receive vector can be formulated as

$$\tilde{\mathbf{y}}_r = \mathbf{F}_N\mathbf{H}_c\mathbf{x}' + \overbrace{\mathbf{F}_N\mathbf{n}}^{\tilde{\mathbf{v}}} \tag{4}$$

$$= \mathbf{F}_N\mathbf{H}_c\mathbf{F}_N^{-1}(\mathbf{G}\tilde{\mathbf{d}} + \tilde{\mathbf{x}}_u) + \tilde{\mathbf{v}}. \tag{5}$$

The matrix $\widetilde{\mathbf{H}} = \mathbf{F}_N\mathbf{H}_c\mathbf{F}_N^{-1}$ is diagonal and contains the sampled channel frequency response on its main diagonal. As a first receiver processing step we subtract the UW influence to obtain the symbol $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_r - \widetilde{\mathbf{H}}\tilde{\mathbf{x}}_u$ and arrive at the form of a linear model

$$\tilde{\mathbf{y}} = \widetilde{\mathbf{H}}\mathbf{G}\tilde{\mathbf{d}} + \tilde{\mathbf{v}}. \tag{6}$$

The channel propagation matrix $\widetilde{\mathbf{H}}\mathbf{G} \in \mathbb{C}^{N\times N_d}$ can be interpreted as the propagation matrix of a complex MIMO channel, although we only modeled a single antenna system. This allows us to treat the UW-OFDM system as a MIMO system with N_d transmit and N receive antennas and to use MIMO detection methods to recover the data.

III. Sphere Decoding

The best decoding results recovering $\tilde{\mathbf{d}}$ from (6) are achieved by a Maximum Likelihood Sequence Estimation on each OFDM symbol. Using the introduced terminology of the linear system model (6) this corresponds to the minimization of the distance of all possible OFDM symbols after channel propagation to the received vector:

$$\hat{\tilde{\mathbf{d}}} = \arg \min_{\tilde{\mathbf{d}}\in\mathcal{A}^{N_d}} ||\widetilde{\mathbf{H}}\mathbf{G}\tilde{\mathbf{d}} - \tilde{\mathbf{y}}||_2^2. \tag{7}$$

In theory, *every* possible data vector $\tilde{\mathbf{d}}$ containing N_d values from the chosen QAM alphabet $\mathcal{A}$ needs to be examined (after symbol generation and channel propagation) for its Euclidean distance to the received signal $\tilde{\mathbf{y}}$. This is impossible for practical UW-OFDM systems. E.g. for the simulation mode used in this work (see Sec. IV-A), there exist $|\mathcal{A}|^{N_d} = 4^{48} > 7.9 \cdot 10^{28}$ possible different OFDM symbols.

In order to allow for MLSE within a practical amount of computation time, Sphere Decoding is a well known method originating from MIMO decoding. We applied the Sphere Decoding algorithm to a single antenna UW-OFDM system [2].

First, a QR decomposition of the transmission matrix enables the required simplifications

$$\widetilde{\mathbf{H}}\mathbf{G} = \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}, \tag{8}$$

where $\mathbf{Q} \in \mathbb{C}^{(N\times N)}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{N_d\times N_d}$ is upper triangular. As the order of QR decomposing the matrix has an impact on the performance of the SD, we use the sorted QR decomposition [6]. Partitioning $\mathbf{Q} = \begin{bmatrix}\mathbf{Q}_1 & \mathbf{Q}_2\end{bmatrix}$, with $\mathbf{Q}_1$ of size $(N \times N_d)$ and $\mathbf{Q}_2$ of size $(N \times N_u)$, we can elaborate the Euclidean distance in (7) as

$$\begin{aligned}\left|\left|\widetilde{\mathbf{H}}\mathbf{G}\tilde{\mathbf{d}}-\tilde{\mathbf{y}}\right|\right|_2^2 &= \left|\left|\mathbf{Q}\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\tilde{\mathbf{d}}-\tilde{\mathbf{y}}\right|\right|_2^2\\ &= \left|\left|\mathbf{Q}^{\mathsf{H}}\mathbf{Q}\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\tilde{\mathbf{d}}-\mathbf{Q}^{\mathsf{H}}\tilde{\mathbf{y}}\right|\right|_2^2\\ &= \left|\left|\mathbf{R}\tilde{\mathbf{d}}-\mathbf{Q}_1^{\mathsf{H}}\tilde{\mathbf{y}}\right|\right|_2^2+\left|\left|\mathbf{Q}_2^{\mathsf{H}}\tilde{\mathbf{y}}\right|\right|_2^2.\end{aligned} \tag{9}$$

The second term is independent of $\tilde{\mathbf{d}}$, and thus the minimization problem (7) becomes

$$\hat{\tilde{\mathbf{d}}} = \arg\min_{\tilde{\mathbf{d}}\in\mathcal{A}^{N_d}} \left|\left|\mathbf{R}\tilde{\mathbf{d}}-\mathbf{Q}_1^{\mathsf{H}}\tilde{\mathbf{y}}\right|\right|_2^2. \tag{10}$$

With the description in (10), one of the many Sphere Decoding algorithms can be applied. For this work we use an algorithm following the Schnorr-Euchner strategy, that was introduced in [7], and adapted towards UW-OFDM [2].

This SD implementation is basically a tree search, looking for the leaf with the smallest l_2-norm (10). However, the transmission circumstances can lead to bad conditions for the SD algorithm, forcing it to check a majority of the possible leaves in order to find the MLSE solution. For practical realizations the maximum detection time must be limited. Using the algorithm in [7] the number of nodes visited is proportional to the execution time of the SD. In [3] it is suggested to limit the number of visited nodes in order to limit the execution time effectively, but preventing the MLSE solution to be found.

The impact of limiting the number of visited nodes to $v_{\max}$ clearly affects the detection performance. The values range from N_d node visits, which returns the leaf found first – this corresponds to the data vector found by successive interference cancellation (SIC) – to the maximum value of $|\mathcal{A}|^{N_d}$, which represents no limitation and always yields the MLSE solution. However, it strongly depends on the particular channel realization and noise values, and – for the case of UW-OFDM – on the OFDM symbol generator matrix $\mathbf{G}$, if the possible receive vectors $\widetilde{\mathbf{H}}\mathbf{G}\tilde{\mathbf{d}}$ are spread far enough apart to impede exhaustive tree searches.

IV. Simulation Results

A. Simulation Setup

A block diagram of the system setup used in this work for simulation is shown in Fig. 1. After the QAM mapping, the OFDM symbol is assembled in frequency domain, transformed and supplied with the UW acc. to (2), resulting in the time domain signal $\mathbf{x}'$. As UW we use the zero word.

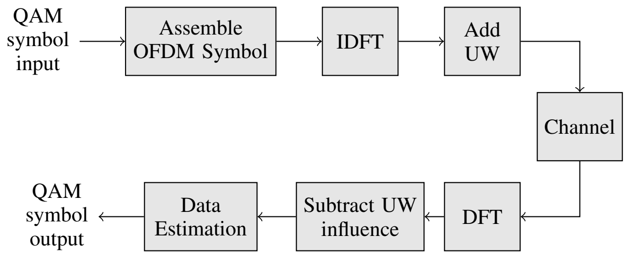

Fig. 1. Block diagram of the transceiver system

Furthermore, we used a system with the parameters shown in Tab. I. The DFT-length as well as the unique word length are chosen to meet the length of the DFT and the guard interval specified in IEEE 802.11a. The indices of the redundant subcarriers were optimized acc. to the criteria formulated in [1].

TABLE I
PARAMETERS OF THE INVESTIGATED UW-OFDM SYSTEM

Modulation scheme		4-QAM
DFT length	N	64
No. of data (red.) subcarriers	N_d (N_r)	48 (16)
Unique Word	$\mathbf{x}_u$	$0^{(N_r\times 1)}$
Indices of red. subcarriers	$\{1,5,9,13,\ldots,57,61\}$	

We compare the Sphere Decoding (SD) performance with the LMMSE estimator, as derived for UW-OFDM in former publications [1]:

$$\mathbf{E}_{\text{LMMSE}} = \left(\mathbf{G}^{\mathsf{H}}\widetilde{\mathbf{H}}^{\mathsf{H}}\widetilde{\mathbf{H}}\mathbf{G}+\frac{N\sigma_n^2}{\sigma_d^2}\mathbf{I}\right)^{-1}\mathbf{G}^{\mathsf{H}}\widetilde{\mathbf{H}}^{\mathsf{H}} \tag{11}$$

B. Simulation Results in the AWGN Channel

Clearly OFDM is not designed for the AWGN channel, but we will point out later that the worst case scenario in terms of this complexity analysis is in fact this one.

In Fig. 2 the BER is shown over E_b/N_0 with different node visit limits, ranging from the minimum of $v_{\max}=48$ to 10 000.

While the minimum node visit limit of $v_{\max}=48$ is the lower performance bound for the SD here (performance of a SIC receiver), the BER performance gains notably with increasing node visit limit. A difference is only notable in higher E_b/N_0 regions, though.

By plotting the BER over the limit of node visits in Fig. 3, we can see an interesting effect more clearly.

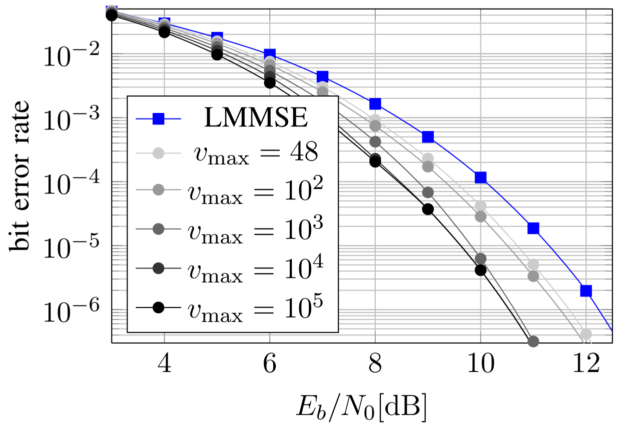

Fig. 2. BER behavior with different node visit limits in the AWGN channel

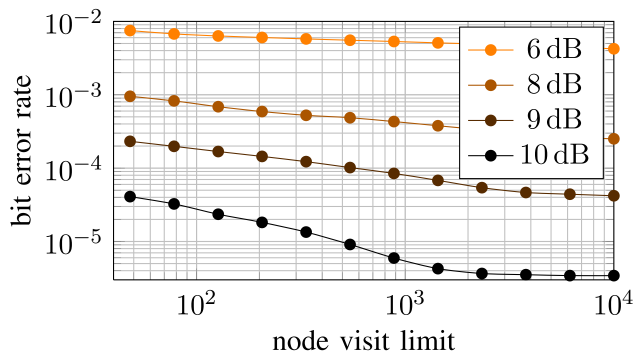

Fig. 3. BER impact of limiting of node visits of the SD in the AWGN channel

As supposed from Fig. 2, increasing the node visit limit at low E_b/N_0 values leads to a very small improvement of the BER only. At higher E_b/N_0, the limit has a rather strong effect on the BER, up to 1 dB. This implies, that with increasing E_b/N_0 the BER margin between the SIC and the MLSE solution becomes larger as well. Interestingly, above a certain node visit limit, the BER almost does not improve anymore, e.g. at $E_b/N_0 = 10\,\text{dB}$ the BER stagnates at about $3.5 \cdot 10^{-6}$ from a node visit limit of about 2 000.

The results in Fig. 3 also suggest the assumption, that at high E_b/N_0 only a small part of the node visit budget is used up, while it is fully spent at low E_b/N_0. The plot of the average number of node visits per OFDM symbol over the maximum allowed in Fig. 4 confirms this pretty well. The curve at $E_b/N_0 = 10\,\text{dB}$ shows no increase of the average number of node visits after the limit of $\sim 2\,000$, from where also the BER did not improve notably. On average about 600 node visits yield the best decoding result. At low E_b/N_0 in contrast, almost the whole node visit quota is utilized for every OFDM symbol.

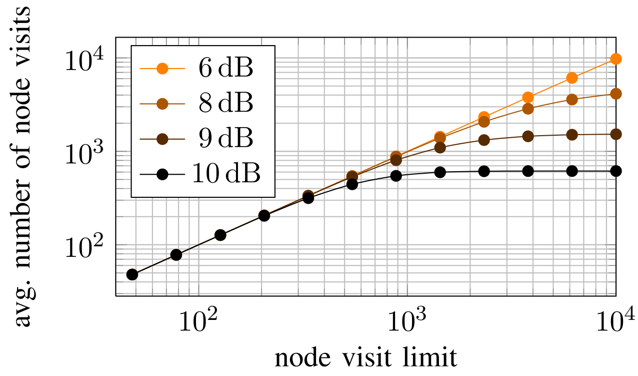

Fig. 4. SNR influence on the average number of nodes visited

The smooth transition from the linear increase of node visits to the almost constant state, compared to a hard break is due to our method of counting node visits: If the limit is reached the SD returns the last best leaf as a decoding result, which was found with fewer node visits, not at the maximum value.

These results demonstrate, that the constrained SD for UW-OFDM tends in the AWGN channel to work the whole SD tree at low E_b/N_0, resulting in very long execution times but providing only a small gain in BER. At high E_b/N_0 the SD terminates itself with high probability after a certain number of node visits, which depends on the actual E_b/N_0 value.

C. Simulation Results in a Multi-Path Channel

OFDM and especially UW-OFDM as well as SD unfold their full potential in a multi-path environment, that is usually more challenging for the receiver. For our run-time analysis however, things improve.

In order to show this, we used a multi-path channel snapshot with 15 taps for the channel impulse response. It's frequency response is shown in Fig. 5. Due to its deep spectral notches it can be considered as a tough environment for communications.

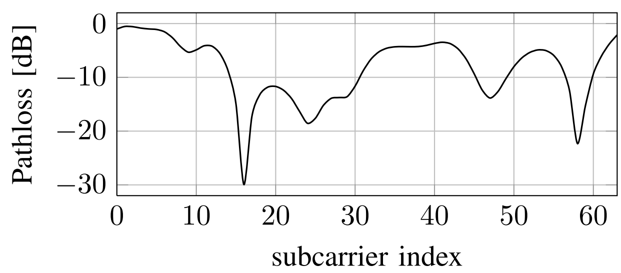

Fig. 5. Frequency response of exemplary multi-path channel

As before, the impact of the node visit limit on the BER in the multi-path channel is shown in Fig. 6.

Fortunately this shows, that by far less node visits are necessary to achieve an SD performance that is close to the MLSE optimum. At low E_b/N_0 the BER still improves with rising limits again, but likewise only in a very small margin, as observed earlier in the AWGN channel.

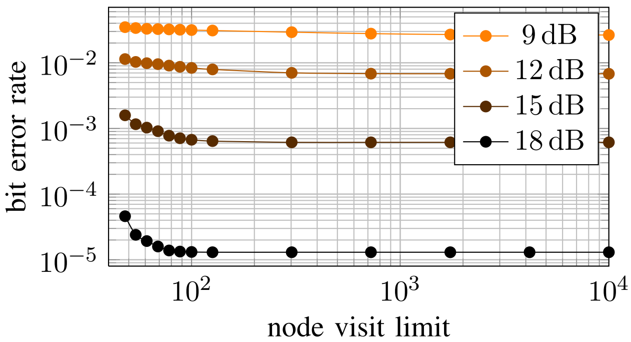

Fig. 6. BER impact of limiting node visits of the SD in the exemplary multi-path channel

Finally, the BER is plotted over E_b/N_0 in Fig. 7, to show the potential of non-linear decoding strategies for UW-OFDM. In a quite corrupting multi-path environment the LMMSE estimator is outperformed by far (already $4\,\mathrm{dB}$ at a BER of 10^{-5}), and by increasing the node visit limit about $1\,\mathrm{dB}$ can be gained over the SIC solution, just as before in the AWGN channel. Additionally, only a small amount of nodes is needed to achieve this performance gain. For the used setup about $v_{\max} = 100$ suffices.

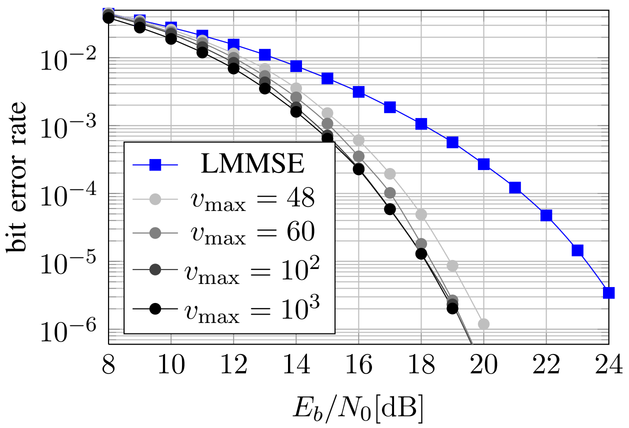

Fig. 7. BER behavior with different node visit limits in the exemplary multi-path channel

V. Conclusion

The distinct system structure of the UW-OFDM signaling scheme allows for all the detection strategies known for MIMO channels, even though considering only a single antenna system. We implemented the Sphere Decoding algorithm for UW-OFDM systems, which is capable of finding the MLSE solution [2]. We verified a huge gain in performance of the Sphere Decoder compared to the LMMSE estimator [1], as it takes the correlations on the redundant subcarriers optimally into account.

However, a major drawback of the SD is, that its run-time complexity varies strongly, making it unsuitable for practical communication systems. To address this, we limited its execution time by introducing an upper bound on the number of nodes, that are visited during the SD tree search [3].

We showed, that at low E_b/N_0 a large part of the SD tree needs to be searched in order to find the MLSE solution, which offers a very small BER improvement only. With increasing E_b/N_0, the SD tends to converge towards the MLSE solution much faster, also providing much higher BER gains. It also became apparent, that strongly disturbing multi-path environments even help the SD to find the MLSE solution more quickly. A flat channel frequency response instead, as for the AWGN channel, cause the computational complexity of the SD to get out of hand very easily.

We conclude, that the UW-OFDM system model together with heavily disturbing multi-path channels proves to be a good environment for implementations of the Sphere Decoder, even when execution time control is necessary for practical reasons.

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